Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

SPACE OF CONSTANT CURVATURE.

BY FREDERICK S. WOODS.

As is well known, the investigation of the foundations of geometry has led, by three distinct methods, to the recognition of the fact that alongside of the Euclidean geometry there exist other systems, equally self consistent and equally capable of explaining the facts of experience. The first of these methods of research, that of Lobachevsky, Bolyai, and Gauss, proceeds upon the historic lines of Euclid. The second method, that of Klein, starts from Cayley's system of measurement with reference to a fundamental quadric and employs the principles of projective geometry. The third method, that of Riemann, consists essentially in the study of differential quadratic expressions and extends to space of any dimensions Gauss's idea of the curvature of a surface. The first two of these methods make less demand than does the third on special mathematical knowledge, and elementary expositions of them are readily accessible and widely known ; but, as far as the present writer is aware, no elementary treatment of Riemann's ideas exists. Such a treatment seems desirable, however, not only because of the intrinsic importance of Riemann's methods, but also because his results have been sometimes misunderstood. The term "curvature of space" has led some to infer by a false analogy that Riemann's geometry is possible only on the assumption of a fourth dimension.*

The present article is an attempt to present Riemann's ideas in an elementary form. The basis of the article is Riemann's famous paper "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen."† In addition, the writer has consulted, among others, the following papers :

Dedekind–Weber: Anmerkungen. Riemann's *Gesammelte Werke*, 1st ed., p. 384; 2nd ed., p. 405.

Helmholtz: Ueber die thatsächlichen Grundlagen der Geometrie (1866). *Wissenschaftliche Abhandlungen*, vol. 2, no. 77, p. 610.

Ueber die Thatsachen, die der Geometrie zum Grunde liegen (1868). *Ibid.*, no. 78, p. 618.

Beltrami: Teoria fondamentale degli spazii di curvatura constante. *Annali di mat.*, ser. 2, vol. 2 (1868), p. 232.

---

* This error has recently been repeated in an article by Professor E. S. Crawley on "Geometry: Ancient and Modern," published in the *Popular Science Monthly* for Jan., 1901, p. 257.

† *Gesammelte Werke*, 1st ed., p. 254; 2nd ed., p. 272.

Saggio di interpretazione della geometria non-Euclidea. *Giornale di mat.*, vol. 6 (1868), p. 284.

Newcomb: Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature .... *Crelle*, vol. 83 (1877), p. 293.

Killing: Ueber zwei Raumformen mit constanter positiver Krümmung; *Crelle*, vol. 86 (1879), p. 72.

Die Rechnung in den Nicht-Euklidischen Raumformen. *Crelle*, vol. 89 (1880), p. 265.

Ueber die Clifford-Klein'schen Raumformen. *Math. Annalen*, vol. 39 (1891), p. 257.

Einführung in die Grundlagen der Geometrie. Padeborn, 1893.

Schur: Ueber die Deformation der Raüme constanten Riemann'schen Krümmungsmasses. *Math. Annalen*, vol. 27 (1886), p. 163.

Ueber den Zusammenhang der Raüme constanten Riemann'schen Krümmungsmasses mit den projectiven Raümen. *Ibid.*, vol. 27 (1886), p. 537.

Klein: *Autographirte Vorlesüngen über Nicht-Euklidische Geometrie.* Göttingen, 1892.

Zur Nicht-Euklidischen Geometrie. *Math. Annalen*, vol. 37 (1890), p. 544.

Lectures on Mathematics, Lecture 11, p. 85, New York, 1894.

Lie-Engel: *Theorie der Transformationsgruppen*, vol. 3, p. 393 ff. Leipzig, 1893.

Bianchi: *Differentialgeometrie*, German translation by Max Lukat, p. 563 ff.

1. The First Two Hypotheses.

Riemann's investigations are concerned with the properties of extents (*Mannigfaltigkeiten*) of any number of dimensions, in which an element may be determined by means of coordinates. Such extents are now commonly called "spaces of n dimensions"; but we shall restrict the word "space", in this article, to an extent of three dimensions, while "surface" shall mean an extent of two dimensions, and "line" an extent of one dimension. We have then two problems. First, we have to develop a system of geometry from assumed hypotheses and, secondly, we have to bring our results to the test of experience and inquire how far they are in accord with natural phenomena.

These two problems need to be sharply distinguished. The first is theoretical and exact; the second is empirical and subject to all the uncertainty which belongs to physical observations. In attacking the first problem, the investigator is free to choose his hypotheses as he will, provided only they are not contradictory, and to draw what conclusions he may from them. We shall accordingly make the two following hypotheses which we will first state and afterwards discuss.

FIRST HYPOTHESIS. *The space to be considered shall be a continuum of three dimensions, in which a point may be determined by three independent real quantities*, x_1, x_2, x_3.

SECOND HYPOTHESIS. *The length of a line shall be determined by means of a line element given by the equation*

$$ds = \sqrt{\Sigma a_{ik}\, dx_i\, dx_k}\,, \qquad (a_{ki} = a_{ik}\,;\ i, k = 1, 2, 3)$$

where the a_{ik} are functions of x_1, x_2, x_3, which together with their differential coefficients of the first four orders are finite and continuous, and where the expression under the radical sign is positive for all real values of x_1, x_2, x_3 and dx_1, dx_2, dx_3, provided that x_1, x_2, x_3 correspond to points of the space and that dx_1, dx_2, dx_3 are not all zero.

Discussion of the First Hypothesis. In a purely analytic sense, a "point" is a set of three values (x_1, x_2, x_3), and " space " is a collection, or assemblage, of such points, which together form a continuum of three dimensions. By the requirement that space be a continuum, the following two properties are asserted of space. First, if (a_1, a_2, a_3) is any point of space, then $(a_1 + h_1, a_2 + h_2, a\ + h_3)$ is also a point of space, whatever may be the values of h_1, h_2, and h_3, provided that no one of them exceeds numerically a certain value h, depending in general on the point (a_1, a_2, a_3). Secondly, if (a_1, a_2, a_3) and (b_1, b_2, b_3) are any two points of space, it is possible to connect them by a continuous curve lying in the space.

Here it is immaterial whether x_1, x_2, and x_3 be allowed to take all possible values, or whether they be subject to certain restrictions, provided only the above two conditions be met.

We assume for the present that to every point of space, or at least to every point of the portion of space which we are studying, corresponds one set of coordinates (x_1, x_2, x_3), and conversely. We can later proceed to all portions of space, in case this condition is not everywhere fulfilled, by the usual methods of analysis.

Our space is assumed to be three dimensional. By this it is not meant to deny the possibility of the space's lying in an extent of higher dimensions, but simply to assert that the discussion has nothing whatever to do with these dimensions. The validity of the results obtained is neither strengthened nor weakened by the assumption of a fourth dimension.

Discussion of the Second Hypothesis. Within the space thus defined we may pick out at pleasure one dimensional extents or lines. We shall restrict ourselves here to lines which may be expressed analytically by the equations

$$x_1 = f_1(t), \qquad x_2 = f_2(t), \qquad x_3 = f_3(t),$$

where t is an arbitrary parameter, and where, for the values of t which come into consideration, f_1, f_2, and f_3 are continuous functions possessing continuous first derivatives which do not vanish simultaneously. Consider now a portion of such a line corresponding to values of t lying between, and including, the values t_0 and T, and let this portion be broken up into n segments, to the extremities of which correspond the values $t_0, t_1, t_2, \ldots\ldots, t_{n-1}, T$. If now (x_1, x_2, x_3) are the coordinates of the extremity $(t = t_i)$ of any segment, the coordinates of the other extremity $(t = t_{i+1})$ will be, except for infinitesimals of higher order, $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$, where

$$dx_1 = f_1'(t)dt, \quad dx_2 = f_2'(t)dt, \quad dx_3 = f_3'(t)dt, \quad \text{and} \quad dt = t_{i+1} - t_i.$$

We introduce now arbitrarily as the definition of an *element of length* a function

$$\phi(x_1, x_2, x_3 ;\ dx_1, dx_2, dx_3)$$

which has the following two properties. First, it shall become infinitesimal with dx_1, dx_2, dx_3 and consequently with dt, and, secondly, the sum of the n values of this function computed for the n segments of the line considered shall approach a limit as n is indefinitely increased and at the same time the n quantities $t_{i+1} - t_i$ are made to approach zero; the limit to be independent of the manner in which the extremities of the segments are chosen. *This limit is defined as the length of the portion of the line in question.*

These conditions are met in particular by the assumption of the line element as in the second hypothesis, in accordance with which we have

$$\phi = \sqrt{\Sigma a_{ik}\, dx_i\, dx_k}.$$

The length of the line is then given by the integral

$$l = \int_{t_0}^{T} \sqrt{\Sigma a_{ik} f_i'(t) f_k'(t)}\ dt.$$

This form of the line element has for our present purpose the decided advantage of including as special cases all the forms occurring in Euclidean geometry, which correspond to different systems of coordinates (Cartesian, polar, cylindrical, etc.) ; but it is by no means the only conceivable one. Thus we might, as pointed out by Riemann, employ the positive fourth root of a homogeneous biquadratic expression in dx_1, dx_2, dx_3. Further, the restriction that the expression $\Sigma a_{ik}\, dx_i\, dx_k$ should be always positive for allowable real values of the quantities involved may be dropped,* but in that case we should have certain real lines with zero or even with imaginary length. Such a geometry would not explain the facts of experience and is therefore not considered here.

2. Geodesic Lines and Surfaces. We proceed now to find the equations of the shortest line between two points; or, in other words, to determine the condition that the integral

$$l = \int_{t_0}^{t_1} \sqrt{\Sigma a_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt}}\, dt$$

shall be a minimum. The Calculus of Variations gives, as a necessary condition that the integral

$$I = \int_{t_0}^{t_1} F(x, y, z, x', y', z')\, dt$$

be a minimum, the Eulerian equations

$$\frac{d}{dt} F_{x'} - F_x = 0, \qquad \frac{d}{dt} F_{y'} - F_y = 0, \qquad \frac{d}{dt} F_{z'} - F_z = 0.$$

In the present case, these become

$$\frac{d}{dt} \frac{1}{\sqrt{R_t}} \left(a_{l1} \frac{dx_1}{dt} + a_{l2} \frac{dx_2}{dt} + a_{l3} \frac{dx_3}{dt} \right) = \frac{1}{2} \frac{1}{\sqrt{R_t}} \sum_{ik} \frac{\partial a_{ik}}{\partial x_l} \frac{dx_i}{dt} \frac{dx_k}{dt},$$

where $l = 1, 2, 3$ and $R_t = \Sigma a_{ik} \dfrac{dx_i}{dt} \dfrac{dx_k}{dt}$. The integral l is invariant of the parameter t. We introduce the parameter s defined by the equation

$$s = \int_{t_0}^{t} \sqrt{\Sigma a_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt}}\, dt,$$

i. e. the length s of the portion of the geodesic line between an assumed fixed point and a variable point. Since

$$\int_{t_0}^{t} \sqrt{R_t}\, dt = \int_{s_0}^{s} \sqrt{R_s}\, ds = s,$$

---

* Compare for example Bianchi's paper: "Sulle forme differenziali quadratiche indefinite," *Atti della R. Accademia dei Lincei*, Memorie, ser. 4, vol. 5 (1888), p. 539, as well as the writer's article: Ueber Pseudominimalflächen, Göttingen 1895, wherein are developed properties of certain surfaces in a space for which the line element is $ds = \sqrt{dx^2 + dy^2 - dz^2}$.

it follows that

$$\Sigma a_{ik} \frac{dx_i}{ds} \frac{dx_k}{ds} = 1.$$

Hence the necessary conditions for a minimum, in terms of the parameter s, are

$$\left.\begin{aligned} \frac{d}{ds}\left[a_{11}\frac{dx_1}{ds} + a_{12}\frac{dx_2}{ds} + a_{13}\frac{dx_3}{ds}\right] &= \tfrac{1}{2}\sum \frac{\partial a_{ik}}{\partial x_1}\frac{dx_i}{ds}\frac{dx_k}{ds}, \\ \frac{d}{ds}\left[a_{12}\frac{dx_1}{ds} + a_{22}\frac{dx_2}{ds} + a_{23}\frac{dx_3}{ds}\right] &= \tfrac{1}{2}\sum \frac{\partial a_{ik}}{\partial x_2}\frac{dx_i}{ds}\frac{dx_k}{ds}, \\ \frac{d}{ds}\left[a_{13}\frac{dx_1}{ds} + a_{23}\frac{dx_2}{ds} + a_{33}\frac{dx_3}{ds}\right] &= \tfrac{1}{2}\sum \frac{\partial a_{ik}}{\partial x_3}\frac{dx_i}{ds}\frac{dx_k}{ds}. \end{aligned}\right\} \qquad (A)$$

Conversely, these conditions are sufficient if l is not too great. More precisely: Let (a_1, a_2, a_3) be any point of space, and (x_1, x_2, x_3) any second point such that $|x_i - a_i|$ does not exceed a suitably chosen positive quantity, h. Then the equations (A) admit one and only one solution which passes through the points (a) and (x) and has all of its points lying in the region $|x_i - a_i| < h$, and for the corresponding curve the integral I has a smaller value than for any other curve joining the points (a) and (x).

We take the equations (A) accordingly, as the defining equations of the *geodesic lines*. In accordance with the theory of differential equations, it is always possible to find one and only one solution of these equations which takes on at an arbitrary point (x_1, x_2, x_3) any arbitrarily assigned values (not all zero) of the differential coefficients dx_1/ds, dx_2/ds, dx_3/ds. If these differential coefficients satisfy initially the relation

$$\Sigma a_{ik} \frac{dx_i}{ds} \frac{dx_k}{ds} = 1,$$

this relation will be fulfilled for all values of s.

The geodesic lines which radiate from a point are hence distinguished from each other by the ratios of the values of the differential coefficients and these may consequently be regarded as fixing the direction of each line; the direction being, broadly, that property of a geodesic line which distinguishes it from all others through the same point.

It will be convenient to denote dx_i/ds by ξ_i, and to define the triple consisting of the three direction quantities ξ_1, ξ_2, ξ_3 as the *direction* (ξ_1, ξ_2, ξ_3). These quantities satisfy the relation

$$\Sigma a_{ik}\, \xi_i\, \xi_k = 1. \qquad (B)$$

Our results thus far are these :

THEOREM 1. *A geodesic line is completely determined by means of any point on it and any second point not too far distant from the first.*

THEOREM 2. *A geodesic line is completely determined by means of any point on it and its direction at that point.*

THEOREM 3. *Through any point of space pass ∞^2 geodesic lines, each of which is completely determined by means of its initial direction.*

Take now two geodesic lines extending from the same point with the directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ and consider the system of geodesic lines which radiate from their point of intersection with the directions (ξ_1, ξ_2, ξ_3), where

$$\begin{aligned} \xi_1 &= \lambda\alpha_1 + \mu\beta_1, \\ \xi_2 &= \lambda\alpha_2 + \mu\beta_2, \\ \xi_3 &= \lambda\alpha_3 + \mu\beta_3, \end{aligned}$$

and, by virtue of the relation (B), λ and μ are connected by the relation

$$\lambda^2 + \mu^2 + 2\lambda\mu\,\Sigma a_{ik}\,\alpha_i\,\beta_k = 1.$$

Such a system we call a *pencil of geodesic lines*, and the common point the *vertex*. This pencil corresponds to the totality of straight lines in elementary Euclidean geometry, all of which go through a common point and lie in the same plane. Any two lines of a pencil define the pencil. Any line of the pencil is fixed by means of a single parameter, which may be conveniently chosen by introducing the conception of an *angle*. *The angle θ between two intersecting geodesic lines, with the directions (ξ_1, ξ_2, ξ_3) and (ξ_1', ξ_2', ξ_3'), shall be defined by the relation*

$$\cos\theta = \Sigma a_{ik}\,\xi_i\,\xi_k',$$

where the a_{ik} are computed for the point of intersection.

The angle θ is always real, since we are considering only real geodesic lines. This may be most conveniently seen by dropping for the moment the condition that (ξ_1, ξ_2, ξ_3) should satisfy relation (B). We may then interpret these quantities as homogeneous point coordinates in a Euclidean plane. If now (ξ_1', ξ_2', ξ_3') and $(\xi_1'', \xi_2'', \xi_3'')$ are two points of this plane, any point of the line connecting them is given by the formula :

$$\xi_1 = \lambda\, \xi_1' + \mu \xi_2'',$$
$$\xi_2 = \lambda\, \xi_1' + \mu \xi_2'',$$
$$\xi_3 = \lambda\, \xi_1' + \mu \xi_2'';$$

and the points in which this line intersects the conic section

$$\Sigma a_{ik}\, \xi_i\, \xi_k = 0$$

correspond to values of $\lambda : \mu$ given by the equation :

$$\lambda^2 \Sigma a_{ik}\, \xi_i'\, \xi_k' + 2 \lambda \mu\, \Sigma a_{ik}\, \xi_i'\, \xi_k'' + \mu^2\, \Sigma a_{ik}\, \xi_i''\, \xi_k'' = 0.$$

But the above conic has no real points of intersection by virtue of our second hypothesis. Therefore the roots of the quadratic in $\lambda : \mu$ are imaginary. Therefore its coefficients satisfy the relation :

$$[\Sigma a_{ik}\, \xi_i'\, \xi_k'] [\Sigma a_{ik}\, \xi_i''\, \xi_k''] > [\Sigma a_{ik}\, \xi_i'\, \xi_k'']^2.$$

Introducing now the condition (B) we have

$$|\Sigma a_{ik}\, \xi_i'\, \xi_k''| < 1.$$

The limiting cases $\Sigma a_{ik}\, \xi_i'\, \xi_k'' = \pm 1$ can occur only when

$$\xi_1' : \xi_2' : \xi_3' = \xi_1'' : \xi_2'' : \xi_3''.$$

Since the signs of ξ_1, ξ_2, and ξ_3 may be changed without altering the geodesic line corresponding to them (for such a change of sign may be made by altering the sign of ds, that is, of the direction in which the length is measured), it follows that $\cos \theta$ is ambiguous in sign. This ambiguity is the same as that met with in the angle between straight lines in Euclidean geometry and may be removed by familiar conventions.

If $\cos \theta = 0$; *i. e.* if

$$\Sigma a_{ik}\, \xi_i\, \xi_k' = 0$$

the two lines are said to be *perpendicular*.

We may now readily prove the following theorems :

THEOREM 4. *In any pencil of geodesic lines there exists one and only one line perpendicular to any given line of the pencil.*

THEOREM 5. *If a geodesic line through the vertex of a pencil is perpendicular to two lines of the pencil, it is perpendicular to all lines of the pencil.*

THEOREM 6. *Through the vertex of a pencil of geodesic lines passes one and only one geodesic line perpendicular to all lines of the pencil.*

THEOREM 7. *The geodesic lines perpendicular to a given geodesic line at a given point form a pencil.*

Proof of Theorem 4. Let a pencil be given with its vertex at O and let OA and OB be any two lines of the pencil with directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ respectively, making the angle ω with each other. Then

$$\cos\omega = \Sigma a_{ik}\,\alpha_i\beta_k.$$

Let OM be any line of the pencil defined by OA and OB; then the direction of OM is given by the equations

$$\xi_i = \lambda\alpha_i + \mu\beta_i,$$

where

$$\lambda^2 + \mu^2 + 2\lambda\mu\cos\omega = 1.$$

The angle θ between OM and OA is given by the formula

$$\cos\theta = \Sigma a_{ik}\,\alpha_i(\lambda\alpha_k + \mu\beta_k) = \lambda + \mu\cos\omega.$$

The condition that OM be perpendicular to OA is that $\cos\theta = 0$; hence we have as the necessary and sufficient condition for perpendicularity:

$$\lambda = \pm\cot\omega, \quad \mu = \mp\csc\omega.$$

The direction of OM is thus restricted to a single direction and its opposite, and hence there is one and only one line in the pencil perpendicular to OA.

Proof of Theorem 5. Let OC be a geodesic line through OA perpendicular to the line OA and OB of the pencil. The direction $(\gamma_1, \gamma_2, \gamma_3)$ of OC satisfies, then, the conditions

$$\Sigma a_{ik}\,\alpha_i\gamma_k = 0, \qquad \Sigma a_{ik}\,\beta_i\gamma_k = 0,$$

and consequently the condition

$$\Sigma a_{ik}(\lambda\alpha_i + \mu\beta_i)\gamma_k = 0$$

for all values of λ and μ. Hence OC is perpendicular to every line of the pencil.

Proof of Theorem 6. By the equations

$$\Sigma a_{ik}\,\alpha_i\gamma_k = 0, \qquad \Sigma a_{ik}\,\beta_i\gamma_k = 0,$$

the values of the ratios of $\gamma_1 : \gamma_2 : \gamma_3$ are uniquely determined, and by aid of (B) the values of these quantities are determined except for sign. Hence it is always possible to determine one and only one perpendicular to two intersecting geodesic lines at their point of intersection. By Theorem 5, this line is perpendicular to every line of the pencil.

Proof of Theorem 7. If OC with direction $(\gamma_1, \gamma_2, \gamma_3)$ be given, the condition that a geodesic line through O should be perpendicular to OC is

$$\Sigma a_{ik} \gamma_i \xi_k = 0.$$

If now $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ are two sets of values which satisfy this equation, it will be satisfied by all values of the form

$$\xi_i = \lambda \alpha_i + \mu \beta_i$$

and by no other values. This follows from the fact that the equation is linear in the unknown quantities ξ_i. Hence the geodesic lines through O perpendicular to OC form a pencil.

Geodesic Surface. If we define a pencil by means of two perpendicular geodesic lines OA and OB, and denote as before by θ the angle (properly chosen) between OA and any line of the pencil OM, we have

$$\lambda = \cos\theta, \qquad \mu = \sin\theta;$$

so that the direction of OM is given by

$$\xi_i = \alpha_i \cos\theta + \beta_i \sin\theta,$$

where θ is arbitrary and $\Sigma a_{ik} \alpha_i \beta_k = 0$.

If $P(x_1, x_2, x_3)$ be any point on the line OM and r be the length of this line from O to P, the coordinates of P are determined by integrating the differential equations of the geodesic line, choosing the solution which at the point x_i has the direction ξ_i, and substituting r for s. We have then

$$x_i = f_i(\xi_1, \xi_2, \xi_3, r) = \phi_i(\theta, r),$$

the functions $\phi_i(\theta, r)$ being continuous, together with their partial derivatives of the first and second orders. By giving to θ and r all possible values, all points of all lines belonging to the pencil are determined. These points, dependent upon two parameters, form an extent of two dimensions which we call a *geodesic surface. A geodesic surface is then defined as a pencil of geodesic lines.* By the definition, the vertex of the pencil forms a unique point of the surface. Whether or not any point of the surface may play this important rôle is at present undecided. We cannot in fact assume that the geodesic surface contains any geodesic lines of space other than those of the pencil.

3. Curvature. *a. The Curvature of a Surface.* Let there be given any surface

$$x_i = \phi_i(u, v), \qquad (i = 1, 2, 3)$$

where u and v are independent variables and the ϕ_i are singled valued continuous functions with continuous differential coefficients of the first two orders. The line element of any line on the surface is found from the line element of space. There results

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where

$$E = \Sigma a_{ik} \frac{\partial x_i}{\partial u} \frac{\partial x_k}{\partial u},$$

$$F = \Sigma a_{ik} \frac{\partial x_i}{\partial u} \frac{\partial x_k}{\partial v},$$

$$G = \Sigma a_{ik} \frac{\partial x_i}{\partial v} \frac{\partial x_k}{\partial v}.$$

The properties of the surface belong to two distinct classes. The properties of the first class are those which may be deduced from a study of the line element just written, where E, F, G are treated as functions of u and v, and no reference is had to their dependence upon x_1, x_2, x_3; while those of the second class depend upon the relations between u, v and x_1, x_2, x_3. For example, if the line element of the surface is

$$ds^2 = du^2 + dv^2,$$

the surface may be a plane, a cylinder, or any developable surface, if it lies in Euclidean space, and may be something quite different in a non-Euclidean space, and the properties of the first class are those common to all these surfaces. The distinction between the two classes of properties is often lost sight of in elementary presentations of the theory of surfaces, but it is of fundamental importance to us, since the properties of the first class maintain their significance whatever may be the nature of the space in which the surface lies. We may accordingly use the results of the ordinary theory of surfaces, in so far as they concern these properties. To obtain the properties of the first class, we assume the line element of the surface, and define length, geodesic lines on the surface, and angle in a manner similar to that just employed for space. Thus the equations of the geodesic lines are

$$\frac{d}{ds}\left[E\frac{du}{ds}+F\frac{dv}{ds}\right]=\tfrac{1}{2}\left[\frac{\partial E}{\partial u}\left(\frac{du}{ds}\right)^2+2\frac{\partial F}{\partial u}\frac{du}{ds}\frac{dv}{ds}+\frac{\partial G}{\partial u}\left(\frac{dv}{ds}\right)^2\right]$$

$$\frac{d}{ds}\left[F\frac{du}{ds}+G\frac{dv}{ds}\right]=\tfrac{1}{2}\left[\frac{\partial E}{\partial v}\left(\frac{du}{ds}\right)^2+2\frac{\partial F}{\partial v}\frac{du}{ds}\frac{dv}{ds}+\frac{\partial G}{\partial v}\left(\frac{dv}{ds}\right)^2\right]$$

and the angle between two geodesics is defined by the formula

$$\cos\theta=$$

$$E\left(\frac{du}{ds}\right)'\left(\frac{du}{ds}\right)''+F\left[\left(\frac{du}{ds}\right)'\left(\frac{dv}{ds}\right)''+\left(\frac{dv}{ds}\right)'\left(\frac{du}{ds}\right)''\right]+G\left(\frac{dv}{ds}\right)'\left(\frac{dv}{ds}\right)''.$$

It should be noticed that, if the surface is considered as lying in a given space, the geodesic lines on the surface are not in general geodesic lines of the space, but lengths and angles are the same whether measured on the surface or in space.

The properties which may be deduced for a surface by the discussion of its line element depend largely upon the quantity called by Gauss the measure of curvature, or simply the *curvature*, of the surface. Under that term we understand a quantity K, defined by the relation

$$K=\frac{1}{2\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\left[\frac{F}{E\sqrt{EG-F^2}}\frac{\partial E}{\partial v}-\frac{1}{\sqrt{EG-F^2}}\frac{\partial G}{\partial u}\right]\right.$$
$$\left.+\frac{\partial}{\partial v}\left[\frac{2}{\sqrt{EG-F^2}}\frac{\partial F}{\partial u}-\frac{1}{\sqrt{EG-F^2}}\frac{\partial E}{\partial v}-\frac{F}{E\sqrt{EG-F^2}}\frac{\partial E}{\partial u}\right]\right\}.$$

With the geometric interpretation of the curvature as usually given on the hypothesis that the surface lies in Euclidean space we have nothing to do. For us, the "curvature" means simply the above expression, which is fully determined when the line element of the surface is given, and may be shown to be an invariant of the surface, that is, independent of the coordinates chosen to define the surface. When K is the same for all points of the surface, the surface is said to be one of *constant curvature*. The importance of the curvature lies in the two theorems:

A necessary condition that two portions of surfaces may be brought into point for point correspondence with preservation of distance (*and hence of angle*) *is that they have the same curvature at corresponding points.*

If the two portions of surfaces are of constant curvature, the condition is also sufficient.

It follows that on any surface of constant curvature there exists a geometry of geodesic lines which is independent of the portion of the surface considered.

Beltrami* has shown that this geometry assumes different distinctive features according as the constant curvature is positive, negative, or zero.

b. The Curvature of Space. As we have seen, the coordinates of a point on a geodesic surface may be taken as (r, θ). The element of arc of a geodesic surface is then

$$ds^2 = E\,dr^2 + 2F\,dr\,d\theta + G\,d\theta^2,$$

where it is possible to determine more exactly the values of the coefficients. In the first place, if $\theta =$ const., $ds = dr$ for all values of r, no matter what value may have been assigned to θ; hence $E = 1$. Secondly, the lines $r = s$, $\theta =$ const. are geodesic lines in space and hence geodesic lines on the surface. Substituting the values $E = 1$, $\frac{d\theta}{ds} = 0$ in the equations of the geodesic lines we have

$$\frac{\partial F}{\partial r} = 0,$$

and therefore F is a function of θ only. When $r = 0$ the quantities x_1, x_2, and x_3 reduce to the coordinates of the vertex of the generating pencil, no matter what the value of θ. Hence

$$\left(\frac{\partial x_1}{\partial \theta}\right)_{r=0} = 0, \qquad \left(\frac{\partial x_2}{\partial \theta}\right)_{r=0} = 0, \qquad \left(\frac{\partial x_3}{\partial \theta}\right)_{r=0} = 0.$$

From the formula

$$F = \Sigma a_{ik} \frac{\partial x_i}{\partial r} \frac{\partial x_k}{\partial \theta},$$

it follows that $(F)_{r=0} = 0$, and hence, since F is a function of θ only, $F \equiv 0$. The line element of the geodesic surface is therefore

$$ds^2 = dr^2 + G\,d\theta^2.$$

---

* **Saggio di interpretationi della geometria non-Euclidea,** ***Giornale di matematiche,*** **vol. 6, (1868), p. 284,** ***cf.*** **also, Bianchi,** ***Differentialgeometrie,*** **p. 418 ff.**

The expression of the curvature reduces in this case to

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial r^2}\,.$$

The curvature is consequently a function not only of r and θ, the coordinates of a point on the surface, but also in general of the coordinates of the vertex O of the generating pencil and of the directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ which fix the pencil, since the functions $\phi_i(r, \theta)$ depend upon these latter quantities. In particular, if K_0 is the curvature at O, it may be shown that

$$K_0 = -\tfrac{1}{2}\,\frac{\sum\limits_{iklm}(ik, lm)_0\,(\alpha_i\beta_k - \alpha_k\beta_i)(\alpha_l\beta_m - \alpha_m\beta_l)}{\sum\limits_{iklm}(a_{il}a_{km} - a_{im}a_{kl})_0\,(\alpha_i\beta_k - \alpha_k\beta_i)(\alpha_l\beta_m - \alpha_m\beta_l)}\,,$$

where

$$(ik, lm) =$$

$$\frac{\partial^2 a_{il}}{\partial x_k \partial x_m} + \frac{\partial^2 a_{km}}{\partial x_i \partial x_l} - \frac{\partial^2 a_{im}}{\partial x_k \partial x_l} - \frac{\partial^2 a_{kl}}{\partial x_i \partial x_m} + \tfrac{1}{2}\sum_{st}\Big(p_{skm}\,p_{til} - p_{sim}\,p_{tkl}\Big)\frac{A_{st}}{a}\,,$$

$$p_{\lambda\mu\nu} = \frac{\partial a_{\lambda\mu}}{\partial x_\nu} + \frac{\partial a_{\lambda\nu}}{\partial x_\mu} - \frac{\partial a_{\mu\nu}}{\partial x_\lambda}\,,$$

A_{st} is the minor corresponding to a_{st} in the determinant

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}\,,$$

the subscript 0 denotes that the expressions to which it is affixed are computed for the point O, and the summation is to be extended over all combinations of the value-pairs i, k and l, m, for which $i < k$, $l < m$.*

The quantity K_0 is called by Riemann the *measure of curvature of the space* in a given point and in a given surface direction. This curvature varies in general from point to point in space, and also in a given point is different for the different geodesic surfaces which pass through the point. In case K_0 is constant, the space is said to be one of *constant curvature*. *A space of constant curvature is accordingly one in which the curvatures of all geodesic surfaces computed for the vertices of the generating pencils are the same.*

---

* *Cf.* Dedekind-Weber, *l. c.* also Schur, *Math. Annalen*, vol. 27 (1886), pp. 542–551. A shorter derivation involving the use of differential invariants and Christoffel's symbolic expressions is given by Bianchi, *l. c.* p. 571 ff.

From the expression for K_0 it follows that the necessary and sufficient condition for a space of constant curvature is that the quantities a_{ik} should satisfy identically the relations

$$(ik, lm) = K_0(a_{il}\, a_{km} - a_{im}\, a_{kl}),$$

where K_0 is a constant.

4. The Third Hypothesis. We shall define a *displacement* as *any transformation by which a continuous portion of space is brought into a continuous point for point correspondence either with itself or with some other portion of space in such a manner that the lengths of corresponding portions of lines are the same.*

Let S be any portion of space in which the coordinates of any point P are (x_1, x_2, x_3), and let S' be a portion of space in which the coordinates of a point P' are (x'_1, x'_2, x'_3). Let the line element in S be denoted by

$$ds = \sqrt{\Sigma a_{ik}\, dx_i\, dx_k}\ ,$$

and the line element in S' by

$$ds' = \sqrt{\Sigma a'_{ik}\, dx'_i\, dx'_k}\ ,$$

where a'_{ik} denotes the value of a_{ik} for (x'_1, x'_2, x'_3). In order that S may be displaced into S', it is necessary and sufficient that

$$ds = ds'$$

by virtue of relations of the form

$$x_i = \phi_i(x'_1, x'_2, x'_3), \qquad (i = 1, 2, 3)$$

where the ϕ_i are continuous functions of (x'_1, x'_2, x'_3), possessing continuous first derivatives, and establishing a one-to-one relation between the points of S and S'.

From this follows

$$a'_{ik} = \sum_{lm} a_{lm} \frac{\partial \phi_l}{\partial x'_i} \frac{\partial \phi_m}{\partial x'_k}.$$

Any displacement transforms geodesic lines into geodesic lines, and geodesic surfaces into geodesic surfaces.

The first part of this theorem follows at once from the definition of geodesic lines. To prove the second part, we notice that

$$\frac{dx_i}{ds} = \sum_l \frac{\partial\phi_i}{\partial x'_l}\frac{dx'_l}{ds'}\frac{ds'}{ds} = \sum_l \frac{\partial\phi_i}{\partial x'_l}\frac{dx'_l}{ds'}.$$

Hence, if $(\alpha_1, \alpha_2, \alpha_3)$ is the direction of a geodesic line at a point O, and $(\alpha'_1, \alpha'_2, \alpha'_3)$ the direction of the corresponding line at O',

$$\alpha_i = \sum_l \left(\frac{\partial\phi_i}{\partial x'_l}\right)_0 \alpha'_l.$$

Since this relation is linear, a pencil of geodesic lines with its vertex at O is transformed into a pencil of geodesics with the vertex at O'.

Under any displacement the angle between two geodesic lines is equal to the corresponding angle between the two corresponding geodesic lines.

For let OA and OB be two geodesic lines intersecting at O with the directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ respectively, and let $O'B'$ and $O'A'$ be the corresponding lines with directions $(\alpha'_1, \alpha'_2, \alpha'_3)$ and $(\beta'_1, \beta'_2, \beta'_3)$. Then if θ is the angle between OA and OB and θ' the angle between OA' and OB',

$$\begin{aligned} \cos\theta &= \sum_{ik} a_{ik}\,\alpha_i\,\beta_k \\ &= \sum_{lm}\left\{\sum_{ik} a_{ik}\,\frac{\partial\phi_i}{\partial x'_l}\,\frac{\partial\phi_k}{\partial x'_m}\right\}\alpha'_l\,\alpha'_m \\ &= \sum_{lm} a'_{lm}\,\alpha'_l\,\alpha'_m = \cos\theta'. \end{aligned}$$

To remove the ambiguity involved in the determination of an angle from its cosine, and at the same time to define corresponding angles, we may consider first the case in which OB coincides with OA. Then $O'B'$ coincides with $O'A'$ and we may take $\theta = \theta' = 0$. Let now OB change its position continuously, the point O being fixed; θ then varies continuously, attaining a final value which we call the angle AOB. At the same time $O'B'$ changes its position continuously, always corresponding to OB, and θ' varies continuously from zero to a final value $A'O'B$, which is the angle corresponding to AOB. Since at all times $\cos\theta = \cos\theta'$, and at the outset $\theta = \theta'$, it follows that $AOB = A'O'B'$.

The possibility of the reversal of the direction of the angle is not excluded, and it seems unnecessary at this point to introduce a distinction between displacements which do not reverse the direction of the angle and those which do.

The existence of displacements is by no means necessary in a general space. This is to be seen by comparing the similar problem in connection with surfaces. A displacement on a surface may be carried out by bending one portion of the surface upon another portion without tearing or stretching. It is clear that surfaces exhibit widely different possibilities in regard to such bending. It may also be seen that the displacement of a portion of a surface and of the entire surface, are different problems. Take, for example, the right circular cylinder in Euclidean space. If a point on the cylinder is fixed, the cylinder as a whole is immovable, but a restricted portion of the cylinder around the fixed point is easily bent upon itself.

The existence of displacements in space is consequently the subject of a new hypothesis.

THIRD HYPOTHESIS. *If P is any point of space, it shall be possible to displace a restricted portion of space surrounding P upon itself in such a manner that any two geodesic lines through P shall correspond to any other two geodesic lines through P, provided only that the two latter lines make the same angle with each other as do the former lines.*

By such a displacement, the point P remains fixed and any two geodesic pencils with their vertices at P may be brought into point for point correspondence with preservation of distance and angle. Hence (*cf.* §3, *a*), the geodesic surfaces formed by the two pencils must have the same curvature at corresponding points and in particular at P. The quantity K_0 of the previous article must therefore be independent of $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$. Now Schur has proved (*l. c.*) that when the curvature of space is constant at each point it does not change as we pass from point to point. Hence K_0 must also be independent of the coordinates of O. We have therefore the theorem:

In order that the third hypothesis shall be met, it is necessary that the space be one of constant curvature.

In the next paragraph we shall show that in a space of constant curvature the hypothesis is always met and that furthermore there exist displacements by which any point P may be transformed into any other point P'.

5. Normal Form of the Line Element. Let the line element

$$ds^2 = \Sigma a_{ik}\, dx_i\, dx_k \tag{1}$$

belong to a space of constant curvature k^2, where k may be zero, real, or pure imaginary. It is then always possible by a change of coordinates to reduce the line element to Riemann's normal form

$$ds^2 = \frac{dy_1^2 + dy_2^2 + dy_3^2}{\left[1 + \frac{k^2}{4}(y_1^2 + y_2^2 + y_3^2)\right]^2}. \tag{2}$$

This we will assume without proof.* If now $k = 0$, the form (2) is as simple as possible. Assuming for the present that k is not zero, we shall proceed to find another form for the line element. For that purpose, we shall first make the substitutions

$$y_1 = \frac{2u}{t+1}, \quad y_2 = \frac{2v}{t+1}, \quad y_3 = \frac{2w}{t+1}, \tag{3}$$

where u, v, w, and t are connected by the identity

$$1 + k^2(u^2 + v^2 + w^2) = t^2. \tag{4}$$

We have then

$$ds^2 = \frac{du^2 + dv^2 + dw^2 - \frac{1}{k^2}dt^2}{t^2} \tag{5}$$

Finally let us introduce x, y, and z defined by the relations

$$\left.\begin{aligned} u &= \frac{1}{k}\tan kx, \\ v &= \frac{1}{k}\sec kx \tan ky, \\ w &= \frac{1}{k}\sec kx \sec ky \tan kz \\ t &= \sec kx \sec ky \sec kz. \end{aligned}\right\} \tag{6}$$

It is readily seen that, if (u_0, v_0, w_0, t_0) is any set of values satisfying (4), and if (b_1, b_2, b_3) and (x_0, y_0, z_0) are the corresponding sets of values of (y_1, y_2, y_3) from (3) and (x, y, z) from (6) respectively then the points (u, v, w, t) that lie in the neighborhood of (u_0, v_0, w_0, t_0) and satisfy (4) are transformed by (3) and (6) respectively in a one-to-one manner and continuously on *all* the points (y_1, y_2, y_3) and (x, y, z) respectively that lie in the neighborhood of (b_1, b_2, b_3) and (x_0, y_0, z_0).

The line element now takes on the *normal form*

$$ds^2 = \cos^2 kz(\cos^2 ky\, dx^2 + dy^2) + dz^2. \tag{7}$$

If k is pure imaginary, the functions in (6) and (7) may be written as hyperbolic functions and the use of imaginaries obviated. We shall avoid duplication of our work, however, by using (6) and (7) as they stand without regard to the reality of k, but it may be noticed that k always appears in such combinations that the resulting expressions are real.

---

* This form is given by Riemann without proof. It is proved by Dedekind-Weber (*l. c.*) on the assumption that the geodesic surfaces are of constant curvature. Since this assumption has been shown by Schur (*l. c.*) to be warranted, the proof is valid.

A proof has also been given by Lipschitz in connection with extended investigations on the properties of differential expressions; *Crelle*, vols. 70, 71, 72 (1869/70).

The differential equations of the geodesic lines (A) become, after a few easy simplifications,

$$\left.\begin{aligned}\frac{d^2x}{ds^2} &= 2k\tan ky\frac{dx}{ds}\frac{dy}{ds} + 2k\tan kz\frac{dx}{ds}\frac{dz}{ds},\\ \frac{d^2y}{ds^2} &= 2k\tan kz\frac{dy}{ds}\frac{dz}{ds} - k\cos ky\sin ky\left(\frac{dx}{ds}\right)^2,\\ \frac{d^2z}{ds^2} &= -k\cos kz\sin kz\left[\cos^2 ky\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right].\end{aligned}\right\} \quad (8)$$

The Coordinates (x, y, z). The coordinates (x, y, z) introduced analytically by equations (3), (4), (6) are analogous to Cartesian coordinates. For, we can see by inspection that all lines of the three types:

(a) $x = \text{const.}$, $y = \text{const.}$, $z = s$;
(b) $x = \text{const.}$, $y = s$, $z = 0$;
(c) $x = s$, $y = 0$, $z = 0$;

satisfy equations (8) and are therefore geodesic lines. In particular, the three lines $(x = s,\ y = 0,\ z = 0)$, $(x = 0,\ y = s,\ z = 0)$ and $(x = 0,\ y = 0,\ z = s)$ form a set of three mutually perpendicular geodesic lines intersecting at O. We will call them OX, OY and OZ respectively. Consider now the surface $z = 0$. We shall have geodesic lines lying on this surface if we place $z = 0$ and take y and x as solutions of the equations

$$\frac{d^2x}{ds^2} = 2k\tan ky\,\frac{dx}{ds}\frac{dy}{ds},$$

$$\frac{d^2y}{ds^2} = -k\cos ky\sin ky\left(\frac{dx}{ds}\right)^2,$$

which result from placing $z = 0$ in (8). These equations, however, are the same as those which we obtain by applying to the surface $z = 0$, in which the line element is

$$ds^2 = \cos^2 ky\, dx^2 + dy^2$$

the equations of the geodesic lines on any surface, as given in §3, *a*. That is, the shortest line which can be drawn on the surface $z = 0$ connecting any two points is also a geodesic line in space. Hence the surface $z = 0$ is a geodesic surface, any point of which can be taken as the vertex of a generating pencil of geodesic lines. In particular the surface is the geodesic surface determined by OX and OY and may properly be called the surface XOY. Similar results may be obtained for the surfaces $y = 0$ and $x = 0$.

Let now $(x_0, y_0, 0)$ be any point on XOY. Any geodesic line on XOY through this point has then the direction $(\alpha_1, \beta_1, 0)$, where $\alpha_1^2 \cos^2 ky_0 + \beta_1^2 = 1$. Any line of type (a) passing through $(x_0, y_0, 0)$ has the direction $(0, 0, 1)$. Hence the lines of type (a) are perpendicular to the pencil of geodesic lines on XOY through the point in which (a) meet XOY, and are therefore perpendicular to XOY. Similarly the lines of type (b) lie in XOY and are perpendicular to OX. The only line of type (c) is OX. It follows then that if we measure along OX the distance x_0, from the point thus reached measure along a geodesic line perpendicular to OX the distance y_0, and from the point thus attained measure along a geodesic perpendicular to XOY the distance z_0 we reach a point with coordinates (x_0, y_0, z_0). Thus every point (x_0, y_0, z_0) lying in the neighborhood of the point $(0, 0, 0)$ can be reached along one of the paths just described, and along but one such path. As a consequence of the foregoing we have the theorem:

Through any point P lying in the neighborhood of the point $x = y = z = 0$ one perpendicular to the geodesic surface XOY can be drawn and, if we require the perpendicular to lie wholly in the neighborhood of the above point, but one. From any such point P lying in the geodesic surface XOY one perpendicular to the geodesic line OX can be drawn and, under the same restriction as before, but one.

To sum up then: *The coordinate system (x, y, z) involves three mutually perpendicular geodesic lines OX, OY, OZ intersecting at O, and determining three geodesic surfaces XOY, XOZ, YOZ. The coordinates of a point P are determined by taking a geodesic line through P perpendicular to XOY meeting it at M, and from M drawing a geodesic line in XOY perpendicular to OX meeting it at N. The lengths of the lines ON, NM, MP are then x, y and z respectively.*

For the sake of symmetry the three geodesic lines and the three surfaces are specified. Practically only one surface XOY, one line OX, and the point O are needed.

6. The Equations of the Geodesic Lines and Surfaces. To obtain the equations of the geodesic lines we need to integrate the differential equations (8), having regard to (7), or its equivalent,

$$1 = \cos^2 ky \cos^2 kz \left(\frac{dx}{ds}\right)^2 + \cos^2 ky \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2. \tag{9}$$

This may be done by considering the functions derived from (6),

$$\left.\begin{aligned} \xi &= \frac{u}{t} = \frac{1}{k} \sin kx \cos ky \cos kz, \\ \eta &= \frac{v}{t} = \frac{1}{k} \sin ky \cos kz, \\ \zeta &= \frac{w}{t} = \frac{1}{k} \sin kz. \end{aligned}\right\} \tag{10}$$

By linear combinations of (8) and (9), we may obtain the equations:

$$\left.\begin{aligned} \frac{d^2\xi}{ds^2} + k^2\xi &= 0, \\ \frac{d^2\eta}{ds^2} + k^2\eta &= 0, \\ \frac{d^2\zeta}{ds^2} + k^2\zeta &= 0. \end{aligned}\right\} \tag{11}$$

To verify this it is only necessary to multiply equations (8) and (9) by the undetermined factors λ, μ, ν, ρ, to add, and to compare the results with equations (11), written out in terms of x, y, z and their differential coefficients. The values of the multipliers are then easily found. *Hence any solution of equations* (8) *which satisfies* (9) *satisfies also* (11).

Conversely, equations (8) may be expressed as linear combinations of (11) and (9). *Hence any solution of* (11) *which satisfies* (9) *also satisfies* (8).

The general solution of (11) is

$$\left.\begin{aligned} \xi &= \frac{A_1}{k} \sin ks + B_1 \cos ks, \\ \eta &= \frac{A_2}{k} \sin ks + B_2 \cos ks, \\ \zeta &= \frac{A_3}{k} \sin ks + B_3 \cos ks. \end{aligned}\right\} \tag{12}$$

It is necessary to choose the constants so as to satisfy (9). The direct substitution is difficult, but we may proceed indirectly as follows. We notice first that (4) gives us, as a relation which t must satisfy, the following:

$$\xi^2 + \eta^2 + \zeta^2 = \frac{1}{k^2} - \frac{1}{k^2 t^2} \tag{13}$$

Furthermore, (9) is equivalent to (5), and the latter may be written:

$$\left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 + \left(\frac{d\zeta}{ds}\right)^2 = 1 - \frac{1}{k^2 t^4}\left(\frac{dt}{ds}\right)^2. \tag{14}$$

Any solution of (11) is also a solution of

$$\xi\frac{d^2\xi}{ds^2} + \eta\frac{d^2\eta}{ds^2} + \zeta\frac{d^2\zeta}{ds^2} + k^2(\xi^2 + \eta^2 + \zeta^2) = 0,$$

which is the same as

$$\frac{1}{2}\frac{d^2}{ds^2}(\xi^2 + \eta^2 + \zeta^2) - \left[\left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 + \left(\frac{d\zeta}{ds}\right)^2\right] + k^2(\xi^2 + \eta^2 + \zeta^2) = 0.$$

Hence we get from (13) and (14), as a necessary condition for t, the equation:

$$\frac{d}{ds}\left(\frac{1}{t}\frac{dt}{ds}\right) = \left(\frac{1}{t}\frac{dt}{ds}\right)^2 + k^2.$$

From this

$$\frac{1}{t}\frac{dt}{ds} = k\tan k(s + a),$$

where a is the constant of integration. No generality is lost by taking $a = 0$, since this amounts to denoting by s the length of the arc measured from a particular point, plus a constant; or simply, as will appear later, to measuring the arc from an arbitrary point. Placing $a = 0$ and integrating the last equation we have

$$t = c\sec ks \tag{15}$$

as a necessary condition for the form of the function t.

It follows from the foregoing that, if the equations (12) are to satisfy (9), they must be consistent with (15). Comparison with (14) gives as necessary relations between the coefficients

$$A_1^2 + A_2^2 + A_3^2 = \frac{1}{c^2} + k^2(B_1^2 + B_2^2 + B_3^2) = 1,$$

$$A_1 B_1 + A_2 B_2 + A_3 B_3 = 0.$$

[TO BE CONTINUED.]

[CONTINUED FROM THE JANUARY NUMBER, PAGE 92.]

Conversely, if the seven constants A, B, and c are merely so chosen as to satisfy these three relations and t is then determined by (15), the functions ξ, η, ζ will satisfy (14) and hence x, y, z will satisfy (9).

By multiplying each of the equations (12) by (15), and placing $A_i c = a_i$, $B_i c = b_i$, we have the theorem:

The equations of any geodesic line may be put into the form

$$\left.\begin{aligned} u &= \frac{a_1}{k}\tan ks + b_1, \\ v &= \frac{a_2}{k}\tan ks + b_2, \\ w &= \frac{a_3}{k}\tan ks + b_3, \\ t &= c \sec ks, \end{aligned}\right\} \tag{16}$$

where the constants are connected by the relations

$$\left.\begin{aligned} a_1^2 + a_2^2 + a_3^2 = 1 + k^2(b_1^2 + b_2^2 + b_3^2) = c^2, \\ a_1 b_1 + a_2 b_2 + a_3 b_3 = 0\,; \end{aligned}\right\} \tag{17}$$

and, conversely, any equations of the form (16) *where the constants satisfy* (17) *represent a geodesic line, if they are satisfied by any points lying in the space under consideration.*

It is important to know that *to any geodesic line corresponds only one set of the above constants, except that all the signs of a_1, a_2, a_3 and c may be changed, and conversely.* For, a geodesic line is completely determined by a point (x_0, y_0, z_0) on it and a direction (α, β, γ), where the signs of all three of the quantities α, β, γ may be changed without altering the line. Then s_0, the value of s corresponding to (x_0, y_0, z_0), is determined by

$$\tan ks_0 = \alpha \tan kx_0 + \beta \tan ky_0 + \gamma \tan kz_0,$$

which is obtained by differentiating the last of equations (6) and (16). The value of s_0 is unique within restricted portions of space. The value of c is then determined by

$$c \sec ks_0 = \sec kx_0 \sec ky_0 \sec kz_0,$$

and the values of a_1, a_2, a_3 are found by differentiating the first three of equations (6) and (16). These latter values depend linearly on α, β, γ. Finally b_1, b_2, b_3 are found from (6) and (16). In this way all the constants are uniquely determined. The converse is proved by noticing that if the equations (16) are given and if, in conjunction with (6), they determine a line lying in the space under consideration, the direction of this geodesic is uniquely determined at any one of its points, and therefore the geodesic is unique. No curve at all results from the equations, however, unless at least one point (x_0, y_0, z_0) within the space considered satisfies the equations.

The equations of the geodesic line are readily written in terms of any point upon it. For, let u_0, v_0, w_0 correspond to any such point and let s_0 be the corresponding value of s. These values determine b_1, b_2, b_3 and the first three of equations (16) become

$$\left.\begin{aligned} u &= \frac{a_1}{k}(\tan ks - \tan ks_0) + u_0, \\ v &= \frac{a_2}{k}(\tan ks - \tan ks_0) + v_0, \\ w &= \frac{a_3}{k}(\tan ks - \tan ks_0) + w_0. \end{aligned}\right\} \tag{19}$$

These are of the form

$$\frac{u - u_0}{c_1} = \frac{v - v_0}{c_2} = \frac{w - w_0}{c_3}.$$

Conversely, any equations of the form

$$\frac{u - u_0}{c_1} = \frac{v - v_0}{c_2} = \frac{w - w_0}{c_3},$$

when u_0, v_0, w_0 correspond to a point (x_0, y_0, z_0) of space, determine a geodesic line.

For, the equations may be written

$$\frac{u - (u_0 + \lambda c_1)}{\rho c_1} = \frac{v - (v_0 + \lambda c_2)}{\rho c_2} = \frac{w - (w_0 + \lambda c_3)}{\rho c_3},$$

where λ and ρ are undetermined multipliers. If we place

$$\begin{aligned} a_1 &= \rho c_1, & b_1 &= u_0 + \lambda c_1, \\ a_2 &= \rho c_2, & b_2 &= v_0 + \lambda c_2, \\ a_3 &= \rho c_3, & b_3 &= w_0 + \lambda c_3, \end{aligned}$$

determine λ and ρ by (17) and substitute the values of a, b thus obtained in (16), we see that the geodesic line thus defined coincides with the given locus.

To obtain the equation of the geodesic surface, consider a pencil of geodesic lines with its vertex at (u_0, v_0, w_0). Since the constants c_1, c_2, c_3 of the last theorem are connected with the coordinates α, β, γ of the direction of the geodesic line by linear relations, the pencil has the equations

$$\frac{u - u_0}{\lambda c_1' + \mu c_1''} = \frac{v - v_0}{\lambda c_2' + \mu c_2''} = \frac{w - w_0}{\lambda c_3' + \mu c_3''},$$

where c_1', c_1'' etc. are constants. The elimination of λ and μ gives a linear equation

$$lu + mv + nw + p = 0.$$

Conversely, let any such linear equation be given, which is satisfied by at least one point (u_0, v_0, w_0) of the space considered. The necessary and sufficient conditions that a geodesic line

$$\frac{u - u_0}{c_1} = \frac{v - v_0}{c_2} = \frac{w - w_0}{c_3}$$

should lie on the corresponding surface are that

$$\begin{aligned} lu_0 + mv_0 + nw_0 + p &= 0, \\ lc_1 + mc_2 + nc_3 &= 0. \end{aligned}$$

The first condition means that the point (u_0, v_0, w_0) lies on the surface. The second is satisfied by

$$c_1 = \lambda c_1' + \mu c_1'', \quad c_2 = \lambda c_2' + \mu c_2'', \quad c_3 = \lambda c_3' + \mu c_3'',$$

where c_1', c_2', c_3' and c_1'', c_2'', c_3'' are any two sets of values which satisfy it, and λ and μ are variable parameters. From any point of the surface, therefore, radiates a pencil of geodesic lines lying wholly on the surface. Moreover, it is easy to see that any point of the surface within a restricted region about the vertex of the pencil is reached by some line of the pencil. To sum up:

Any geodesic surface is represented by a linear equation in u, v, w; and, conversely, any linear equation in u, v, w represents a geodesic surface, if it represents anything at all in space.

Any point of a geodesic surface may be taken as the vertex of a generating pencil of geodesic lines.

The forms of the equations of geodesic lines and surfaces are the same as those of straight lines and planes in Euclidean geometry. They have, therefore, in common with straight lines and planes those properties which may be deduced from the forms of the equations. In particular we note the following:

THEOREM 8. *Any three points not in the same geodesic line and lying in a restricted portion of space determine a geodesic surface.*

THEOREM 9. *If two points on a geodesic surface are connected by a geodesic line, the line lies wholly on the surface.*

THEOREM 10. *Any two geodesic surfaces intersect in a geodesic line if they intersect at all.*

7. Displacements. We shall now show that our space satisfies the third hypothesis.

For that purpose it will be convenient to introduce an auxiliary space, which shall be Euclidean in its properties and in which u, v, and w shall be the rectangular Cartesian coordinates of a point. This auxiliary space we shall denote by S', while the space of constant curvature, whose properties we are investigating, we shall call S. The quantities u, v, w have then a double interpretation; on the one hand, in S, they are defined by (6), on the other hand in S' they are ordinary Cartesian coordinates. We thus establish a point for point correspondence between S and S' by which geodesic lines and surfaces in S correspond respectively to straight lines and planes in S'.

We proceed now to ask what corresponds in S' to the geodesic distance between two points in S. For that purpose consider the equation of a geodesic line in the form (16) and let A and B be two points upon it, to which correspond respectively the values s_1, u_1, v_1, w_1 and s_2, u_2, v_2, w_2. Then the geodesic distance between A and B is $s_2 - s_1$. From the equations of the geodesic line, with regard to the relations (17) which connect the constants, we readily compute

$$k^2(u_1u_2 + v_1v_2 + w_1w_2) + 1 = c^2(\tan ks_1 \tan ks_2 + 1)$$

$$k^2(u_1^2 + v_1^2 + w_1^2) + 1 = c^2 \sec^2 ks_1,$$

$$k^2(u_2^2 + v_2^2 + w_2^2) + 1 = c^2 \sec^2 ks_2,$$

whence follows

$$\cos(s_2 - s_1) = \frac{k^2(u_1u_2 + v_1v_2 + w_1w_2) + 1}{\sqrt{k^2(u_1^2 + v_1^2 + w_1^2) + 1}\ \sqrt{k^2(u_2^2 + v_2^2 + w_2^2) + 1}}$$

This formula which we have obtained in S has a good significance in S'. For let us take in S' the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

as the fundamental quadric in a Cayley system of projective measurement. Then, if M is the distance between two points (u_1, v_1, w_1) and (u_2, v_2, w_2), we have, in accordance with a well known formula of projective geometry,

$$\cos M = \frac{k^2(u_1u_2 + v_1v_2 + w_1w_2) + 1}{\sqrt{k^2(u_1^2 + v_1^2 + w_1^2) + 1}\ \sqrt{k^2(u_2^2 + v_2^2 + w_2^2) + 1}}$$

and hence

$$M = s_2 - s_1.$$

Hence we may say: *If we adopt in S' a system of projective measurement referred to the quadric surface*

$$k^2(u^2 + v^2 + w^2) + 1 = 0,$$

then the geodesic distance between two points in S is equal to the projective distance between the two corresponding points in S'.

Now in order that we may obtain a displacement in S it is necessary and sufficient that we find a continuous one-to-one correspondence between the points of S by means of which geodesic lines go into geodesic lines and the length of any portion of a geodesic line is unchanged. That these conditions are necessary was shown in §4; that they are sufficient may be shown by considering the line element as the length of a geodesic line between two indefinitely near points (x, y, z) and $(x + dx, y + dy, z + dz)$.

To any displacement in S corresponds accordingly a transformation in S' by which straight lines go into straight lines and the projective length of any portion of a straight line is unchanged.

The projective geometry teaches that the transformations with these properties are the collineations of which the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is an invariant. Accordingly we may say:

The displacements in S correspond to the collineations in S' of which

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is an invariant; and conversely.

These collineations are readily written down. They are

$$\left.\begin{aligned} u' &= \frac{\alpha_1 u + \alpha_2 v + \alpha_3 w + \alpha_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \\ v' &= \frac{\beta_1 u + \beta_2 v + \beta_3 w + \beta_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \\ w' &= \frac{\gamma_1 u + \gamma_2 v + \gamma_3 w + \gamma_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \end{aligned}\right\} \qquad (18)$$

where $\alpha_1 \beta_2 \gamma_3 \delta_4 | \neq 0$, and the sixteen constants are connected by the ten conditions:

$$\left.\begin{aligned} k^2(\alpha_i^2 + \beta_i^2 + \gamma_i^2) + \delta_i^2 &= k^2, \qquad (i = 1, 2, 3) \\ k^2(\alpha_4^2 + \beta_4^2 + \gamma_4^2) + \delta_4^2 &= 1, \\ \alpha_i \alpha_h + \beta_i \beta_h + \gamma_i \gamma_h + \delta_i \delta_h &= 0. \qquad (i, h = 1, 2, 3, 4;\ i \neq h) \end{aligned}\right\} \qquad (19)$$

We now need to show that the displacements (18) satisfy the demands of the third hypothesis. To prove this directly we might show first that the angle between two geodesic lines in S is equal to the angle between the corresponding lines in S', provided the latter angle is measured with reference to the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0.$$

We might then show directly that the third hypothesis is satisfied in S', and hence in S. But this method would demand extended reckoning. We shall choose an indirect method and shall show first that the third hypothesis is satisfied in a particular point, the origin, and shall then show that it is consequently satisfied in any point of space.

Consider then the origin. The origin is unique in our system of coordinates because there, and there only, the angle between two intersecting geodesic lines in S is equal to the Euclidean angle between the two corresponding straight lines in S'.

To show this, consider a geodesic line in the form (16). If this line goes through the origin with the direction (ξ_1, ξ_2, ξ_3), we have by applying the methods of p. 93,

$$s_0 = 0, \ b_1 = b_2 = b_3 = 0,$$
$$a_1 = \xi_1, \ a_2 = \xi_2, \ a_3 = \xi_3, \ c = \pm 1.$$

The equation of §2 for the angle between two such lines becomes

$$\cos\theta = \xi_1 \xi_1' + \xi_2 \xi_2' + \xi_3 \xi_3'$$
$$= a_1 a_1' + a_2 a_2' + a_3 a_3'.$$

The corresponding straight lines in S' are

$$\frac{u}{a_1} = \frac{v}{a_2} = \frac{w}{a_3}$$

and

$$\frac{u}{a_1'} = \frac{v}{a_2'} = \frac{w}{a_3'},$$

whence, if ϕ is the Euclidean angle between them,

$$\cos\phi = a_1 a_1' + a_2 a_2' + a_3 a_3'$$
$$= \cos\theta.$$

With proper conventions as to corresponding angles, we have

$$\phi = \theta.$$

Consider now a displacement in S for which the origin is a fixed point. From (18) and (19), it will correspond in S' to a collineation of the type

$$\left.\begin{aligned} u' &= \alpha_1 u + \alpha_2 v + \alpha_3 w, \\ v' &= \beta_1 u + \beta_2 v + \beta_3 w, \\ w' &= \gamma_1 u + \gamma_2 v + \gamma_3 w, \end{aligned}\right\} \quad (R)$$

where

$$\alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1, \qquad (i = 1, 2, 3)$$
$$\alpha_i \alpha_h + \beta_i \beta_h + \gamma_i \gamma_h = 0. \qquad (i, h = 1, 2, 3;\ i \neq h)$$

This is a Euclidean rotation about the origin, or a rotation combined with a reflection on one of the coordinate planes, according as $|\alpha_1 \beta_2 \gamma_3| = \pm 1$. We shall use the rotation only and shall denote it by (R). Let now OK and

OL be any two geodesic lines in S intersecting at the origin and let OM and ON be any two other lines, such that the angle KOL equals the angle MON. We wish to show that there exists a displacement by which OK becomes OM and OL becomes ON.

In S' we have four corresponding straight lines $O'K'$, $O'L'$, $O'M'$, $O'N'$ such that the Euclidean angle $K'O'L'$ equals the angle $M'O'N'$. Now there exists in S' a rotation (R) by which $O'K'$ may be brought into coincidence with $O'M'$ and $O'L'$ with $O'N'$. This is a well known fact which of course has its analytic proof. Hence we conclude that the required displacement exists in S.

The third hypothesis is accordingly satisfied at the origin.

We shall now show that the hypothesis is satisfied at any point of space. Let P be any point of S and PK, PL, PM, PN four geodesic lines such that the angle KPL equals the angle MPN. We wish to show that there exists a transformation of type (18) by which PK goes into PM and PL into PN. Among the transformations (18) there are ∞^3 transformations by means of which P goes into O. Take any particular one of these, no matter which, and call it T. Let PK, PL, PM and PN become OK_1, OL_1, OM_1 and ON_1 by means of T. Then the angle K_1OL_1 equals the angle M_1ON_1, since all displacements preserve angles. Then there exists, as we have seen, a displacement R_1 by which OK_1 goes into OM_1, and OL_1 into ON_1. Let T^{-1} be the inverse transformation of T. Then, if we carry out in order the transformations T, R_1, and T^{-1}, we have a transformation which transforms PK into PM and PL into PN. Moreover, this transformation is of type (18). This follows geometrically from the fact that all the transformations we have used are collineations of which

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is invariant, and (18) is the general form of such a collineation.

Hence *the third hypothesis is satisfied in any space of constant curvature.*

We are accordingly justified in proving general propositions by taking the lines involved in any convenient position. We may, for example, apply to any geodesic line, a result already proved for the line OX and may state the following theorem:

THEOREM 11. *Within a properly restricted portion of a geodesic surface, one and only one geodesic line can be drawn perpendicular to a given geodesic line from a point outside of it.*

Corollary. Within a properly restricted portion of space, one and only one perpendicular can be drawn to a geodesic surface from a point outside of it.

8. Space of Zero Curvature. If $k = 0$, Riemann's form of the line element is

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The coordinates of the direction (ξ_1, ξ_2, ξ_3) of a geodesic line are connected by the relation

$$\xi_1 + \xi_2^2 + \xi_3^2 = 1$$

and the angle between two geodesic lines is given by the formula

$$\cos\theta = \xi_1 \xi_1' + \xi_2 \xi_2' + \xi_3 \xi_3'.$$

The differential equations of the geodesic lines are

$$\frac{d^2x}{ds^2} = 0, \qquad \frac{d^2y}{ds^2} = 0, \qquad \frac{d^2z}{ds^2} = 0.$$

The general equations of the geodesic lines are, therefore,

$$\frac{x - b_1}{a_1} = \frac{y - b_2}{a_2} = \frac{z - b_3}{a_3} = s$$

where $$a_1^2 + a_2^2 + a_3^2 = 1.$$

The equation of the geodesic surfaces is

$$lx + my + nz + p = 0.$$

It is evident that the coordinates x, y and z are equivalent to u, v and w of the previous paragraphs, and that Theorems 1–11 still hold. In addition we may deduce a theorem relating to the angles of a triangle which is peculiar to a space of zero curvature. Before stating the theorem, however, we need to make the conceptions of a triangle and of the angles of a triangle precise.

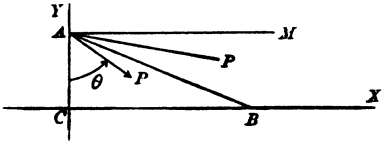
If we select, in the region in which Theorems 1–11 hold, any three points not in the same geodesic line, we may connect them in pairs by geodesic lines and pass a geodesic surface through them. The geodesic lines lie then on this geodesic surface, and the portion of the latter bounded by them we call a *triangle.* The triangle divides the points of the surface not lying on its sides into two classes, *interior* and *exterior* points of the triangle.

To remove the ambiguity inherent in the determination of an angle from the formula above given, we may make the following convention as to the angles of a triangle. If the vertices of the triangle are A, B, and C, the angle at A shall be determined by considering the geodesic lines which radiate from A and enter the triangle ABC. If now a variable line is considered to start from coincidence with AB and take successively the positions of all these lines until it coincides with AC, the angle θ made by the varying line with AB varies continuously from zero to a final value A, which we call the *angle* A of the triangle. We may now state the theorem:

THEOREM 12. *Within a properly restricted region of a space of zero curvature, the sum of the angles of any triangle equals* π.

It will be sufficient to prove this theorem for right triangles, for any triangle may be divided into two right triangles by dropping a perpendicular from any vertex to the opposite side, and if the theorem is true for the two right triangles thus formed, it is evidently true for the original triangle.

Consider the geodesic surface $z = 0$, and a point A of which the (xyz)-coordinates are $(0, p, 0)$, where p is positive. Let A and $C(0, 0, 0)$ determine a geodesic line, on which the direction AC is that of the increasing s. Then the direction of AC is $(\alpha_1 = 0,\ \alpha_2 = -1,\ \alpha_3 = 0)$.



Consider now a moving geodesic turning about A continuously in one direction, thereby fixing the angle θ between AC and AP. By Theorem 1, this line will meet CX at the beginning of its motion. For convenience we take the direction of motion to be such that AP meets CX on the positive side of the origin. Let AM be the first position in which the moving geodesic is perpendicular to AC. If the direction of the increasing s is always taken from A, the direction of AM is $(\beta_1 = 1,\ \beta_2 = 0,\ \beta_3 = 0)$. Then the direction of AP is

$$\xi_1 = \alpha_1 \cos\theta + \beta_1 \sin\theta = \sin\theta,$$
$$\xi_2 = \alpha_2 \cos\theta + \beta_2 \sin\theta = -\cos\theta,$$
$$\xi_3 = \alpha_3 \cos\theta + \beta_3 \sin\theta = 0,$$

and hence the equations of AP are

$$x = s \sin \theta,$$
$$y = -s \cos \theta + p,$$
$$z = 0.$$

Let now AB be a line AP which meets CX, and let B be $(0, q, 0)$, where, by the conventions made, q is positive. If we denote by A the angle CAB, we have from the equations of the line

$$\tan A = \frac{q}{p}.$$

Similarly we may show that

$$\tan B = \frac{p}{q},$$

and hence

$$\cot B = \tan A.$$

Since p and q are both positive, and A and B are both acute angles,

$$B = \frac{\pi}{2} - A.$$

The angle $C = \frac{\pi}{2}$, and hence we have $A + B + C = \pi$, as was to be proved.

As has been repeatedly said, the theorems so far obtained have been shown to be valid only in a region in which there exists a one-to-one relation between the points of space and the coordinates (x, y, z). It is, however, possible to demand that this relation exist for the entire space and all possible values of the coordinates. In this case we have

EUCLIDEAN SPACE.

Here all the theorems so far deduced hold without restriction, and we have in addition two others of interest.

THEOREM 13. *In Euclidean space every geodesic line is of infinite length.*

This follows at once from the form of the equations.

THEOREM 14. *On any geodesic surface in Euclidean space, one and only one geodesic line can be drawn through a given point parallel to a given geodesic line; any other geodesic line through the point intersects the given geodesic.*

We define *parallel lines* as follows: A geodesic line through a given point is said to be parallel to a given geodesic line, when it is the limit approached by a geodesic line through the given point intersecting the given geodesic, as the point of intersection becomes indefinitely remote.

The truth of the theorem then follows from the equations used in proving Theorem (12). We had there

$$\tan A = \frac{q}{p}.$$

As the point B recedes indefinitely in either direction, the angle A approaches $\frac{\pi}{2}$, and the parallel line is AM. Any other value of A determines q when p is fixed.

These fourteen theorems form the groundwork of Euclidean geometry.

Euclidean space is not however the only conceivable space of zero curvature. A systematic investigation of all possible cases seems to be yet lacking, although the general lines on which such an investigation would proceed have been marked out by Klein and some of the problems have been attacked by Killing.* We will content ourselves with mentioning three possibilities.

First, we may suppose that to all points with coordinates $(x + n_1 c_1, y + n_2 c_2, z + n_3 c_3)$ where c_1, c_2, c_3 are constants, and n_1, n_2, n_3 are any positive or negative integers, shall correspond the same point of space. Such a conception presents no difficulty, at least analytically. Our space is then represented point for point by the interior of a rectangular parallelopiped in Euclidean space, the lengths of the edges of the parallelopiped being c_1, c_2 and c_3 respectively and the opposite faces corresponding point for point with one another. For any portion of space lying within this parallelopiped, Theorems 1–12 hold, but Theorems 13 and 14 are not always true. For, the lines ($x =$ const., $y =$ const.), ($y =$ const., $z =$ const.) and ($z =$ const., $x =$ const.) are certainly finite in length, and for these lines the theorem of parallels ceases to hold. Moreover triangles may be constructed within the entire space thus defined for which Theorem 12 is not true.

Two other examples of spaces of zero curvature may be derived from the preceding by considering that one or two of the edges of the parallelopiped are indefinite in extent.

9. Space of Constant Negative Curvature. If $k = i/R$ where R is a real quantity, the space is one of constant negative curvature, $-1/R^2$.

---

* Klein, *Math. Annalen*, vol. 37 (1890) p. 544.
Killing, *ibid.* vol. 39 (1891) p. 257, and *Grundlagen der Geometrie*, vol. 1, chap. 4.

The element of arc may be written

$$ds^2 = \cosh^2\frac{z}{R}\cosh^2\frac{y}{R}\,dx^2 + \cosh^2\frac{z}{R}\,dy^2 + dz^2,$$

and the functions u, v, w, t are

$$u = R\tanh\frac{x}{R},$$

$$v = R\,\mathrm{sech}\,\frac{x}{R}\tanh\frac{y}{R},$$

$$w = R\,\mathrm{sech}\,\frac{x}{R}\,\mathrm{sech}\,\frac{y}{R}\tanh\frac{z}{R},$$

$$t = \mathrm{sech}\,\frac{x}{R}\,\mathrm{sech}\,\frac{y}{R}\,\mathrm{sech}\,\frac{z}{R}.$$

To any set of real values of (x, y, z) correspond then one and only one real set of values of u, v, w, t. Since, however, $\mathrm{sech}\,\theta$ is numerically not greater than unity for real values of θ, t is numerically less than unity. That is

$$0 \leqq t^2 \leqq 1$$

and consequently

$$0 \leqq u^2 + v^2 + w^2 \leqq R^2$$

for real values of x, y, and z. Conversely, it is readily proved that to any real values of u, v, w satisfying the above conditions correspond one and only one set of values of x, y, and z.

The angles of a triangle obey the following theorem:

THEOREM 12′. *In a properly restricted portion of a space of constant negative curvature, the sum of the angles of any triangle is less than π.*

We may employ the construction and figure of the previous paragraph, but the analytic work will be different. The direction of the line AP must satisfy the relations,

$$\xi_1^2\cosh^2\frac{p}{R} + \xi_2^2 = 1, \qquad \xi_3 = 0.$$

In particular, the direction of AC is $(\alpha_1 = 0,\ \alpha_2 = -1,\ \alpha_3 = 0)$ and of AM $(\beta_1 = \mathrm{sech}\,\frac{p}{R},\ \beta_2 = 0,\ \beta_3 = 0)$. Hence

$$\xi_1 = \sin\theta\,\mathrm{sech}\,\frac{p}{R}, \quad \xi_2 = -\cos\theta, \quad \xi_3 = 0.$$

By (19), §6 the equations of AP are

$$\left.\begin{aligned} u &= R\tanh\frac{x}{R} = a_1R\left[\tanh\frac{s}{R} - \tanh\frac{s_0}{R}\right],\\ v &= R\operatorname{sech}\frac{x}{R}\tanh\frac{y}{R} = a_2R\left[\tanh\frac{s}{R} - \tanh\frac{s_0}{R}\right] + R\tanh\frac{p}{R},\\ &z = 0, \end{aligned}\right\}$$

where s_0 is the value of s for the point A. From (16), §6,

$$\operatorname{sech}\frac{p}{R} = c\operatorname{sech}\frac{s_0}{R}.$$

Hence, by differentiating the equations of AP and substituting the coordinates of A,

$$\xi_1 = a_1\operatorname{sech}^2\frac{s_0}{R} = \frac{a_1}{c^2}\operatorname{sech}^2\frac{p}{R},$$

$$\operatorname{sech}^2\frac{p}{R}\,\xi_2 = a_2\operatorname{sech}^2\frac{s_0}{R} = \frac{a_2}{c^2}\operatorname{sech}^2\frac{p}{R}.$$

Therefore

$$a_1 = c^2\sin\theta\cosh\frac{p}{R},$$

$$a_2 = -c^2\cos\theta,$$

and the equations of AP may therefore be written

$$u = -\tan\theta\cosh\frac{p}{R}\left(v - R\tanh\frac{p}{R}\right).$$

The line AB is satisfied by placing $x = q$, $y = 0$, $\theta = A$; whence

$$\tan A = \frac{\tanh\frac{q}{R}}{\sinh\frac{p}{R}}.$$

Similarly

$$\tan B = \frac{\tanh\frac{p}{R}}{\sinh\frac{q}{R}},$$

and hence

$$\tan A\tan B = \frac{1}{\cosh\frac{p}{R}\cosh\frac{q}{R}}.$$

Since p and q are both positive, $A < \frac{\pi}{2}$, $B < \frac{\pi}{2}$, and $\tan A \tan B < 1$; hence

$$A + B < \frac{\pi}{2}$$

and

$$A + B + C < \pi.$$

The theorem therefore holds for right triangles and consequently for all triangles.

We obtain the simplest example of space of constant negative curvature in assuming that a one-to-one correspondence exists between our entire space and the (x, y, z) coordinates. We have then the

SPACE OF LOBACHEVSKY, BOLYAI AND GAUSS.

Within this space, Theorems 1–11 hold universally. In addition we have the following theorems:

THEOREM 13′. *All geodesic lines are infinite in length.*

This follows at once from the equations.

THEOREM 14′. *On any geodesic surface, two geodesic lines can be drawn through a given point parallel to a geodesic line; all other geodesic lines through the given point are separated by the two parallel lines into two sets, one set consisting of lines which intersect the given geodesic line, the other of lines which do not.*

The definition of parallels remains the same as in Euclidean space. The truth of the theorem follows from the equation

$$u = -\tan\theta \cosh\frac{p}{R}\left(v - R\tanh\frac{p}{R}\right)$$

deduced above for a line through $(x = 0,\ y = p,\ z = 0)$ in the plane $z = 0$. This line intersects OX, or $v = 0$, in the point

$$u = R\sinh\frac{p}{R}\tan\theta, \qquad v = 0, \qquad w = 0.$$

These values correspond to real, infinite, or imaginary values of x, y, z according as

$$R^2 \sinh^2\frac{p}{R}\tan^2\theta <, =, \text{ or } > R^2.$$

Let a be the smallest value of θ for which

$$R^2 \sinh^2 \frac{p}{R} \tan^2 \theta = R^2 ;$$

then
$$\cot a = \sinh \frac{p}{R} = \frac{e^{\frac{p}{R}} - e^{-\frac{p}{R}}}{2}$$

whence
$$e^{\frac{p}{R}} = \cot \tfrac{1}{2} a ,$$

$$a = 2 \cot^{-1} e^{\frac{p}{R}}$$

It is now clear that if $0 \leqq \theta < a$, the geodesic line intersects the line OX, the point of intersection receding indefinitely as θ approaches a; if $a < \theta < \pi - a$, the geodesic line does not intersect OX; if $\pi - a < \theta \gtreqless \pi$, the geodesic again intersects OX. This establishes the theorem.

We have thus the groundwork of Lobachevsky's geometry.*

Klein† has pointed out that other examples of space of constant curvature may be found by considering the polyedrons used by Poincaré in his investigations of the Klein groups. The details of the work are yet to be carried out.

10. Space of Constant Positive Curvature. If $k = 1/R$, where R is a real quantity, the space is one of constant positive curvature $1/R^2$. The functions u, v, w, t are then

$$u = R \tan \frac{x}{R} ,$$

$$v = R \sec \frac{x}{R} \tan \frac{y}{R} ,$$

$$w = R \sec \frac{x}{R} \sec \frac{y}{R} \tan \frac{z}{R} ,$$

$$t = \sec \frac{x}{R} \sec \frac{y}{R} \sec \frac{z}{R} ,$$

It appears that u, v, w, t are single valued functions of x, y, z, but the converse is not true.

If we limit ourselves to such a portion of space that a one-to-one relation exists between its points and the coordinates (u, v, w) and (x, y, z), Theorems 1–11 of the preceding paragraphs hold. In addition we have the following theorem:

---

* *Cf. Zwei geometrische Abhandlungen*, German translation by Engel, Leipzig, 1899.

† *Math. Annalen*, vol. 37 (1890), p. 544.

THEOREM 12″. *In a properly restricted portion of a space of constant positive curvature, the sum of the angles of any triangle is greater than π.*

The proof runs exactly parallel with the proof of the corresponding theorem in space of constant negative curvature.

We obtain an example of a space of constant positive curvature, if we assume that a one-to-one correspondence exists between the points of space and all the triples of values (u, v, w). We have then

SIMPLE SPACE OF CONSTANT POSITIVE CURVATURE.

Within this space, the following distinctive theorems are derived from the equations of the geodesic lines:

Every geodesic line is of constant finite length πR.

All geodesic lines intersecting in a point return again to their point of intersection, and have besides no point in common.

There are no parallel lines in this space.

Another example of a space of constant curvature is obtained if we assume that a one-to-one correspondence exists between the points of space and the sets of four values (u, v, w, t) where

$$1 + R^{-2}(u^2 + v^2 + w^2) = t^2.$$

We have then the

DOUBLE SPACE OF CONSTANT POSITIVE CURVATURE.

Within this space the distinctive theorems hold:

Every geodesic line is of constant finite length $2\pi R$.

All geodesic lines intersecting in a point intersect again in a point at a distance πR from the point of intersection and then return to the point of intersection.

Parallel lines do not exist in this space either.

In the double-space the geometry on the geodesic surfaces is identical with that on the surface of a sphere in Euclidean space. In the simple space the geometry on a geodesic surface is that which would be obtained in Euclidean space by the projection of the surface of the sphere from its centre upon a plane.

For further discussion, the reader is referred to the last cited work of Killing.

11. Relation of our Results to the Facts of Experience. There remains yet the problem to enquire how far the foregoing systems of geometry explain the facts of experience. To deal with this question exhaustively would require more space than is at our disposal. We must content ourselves with a mere outline of the treatment.

We do not wish to discuss the metaphysical questions involved. To some minds the space conceptions which underlie Euclid's work, including his theories of parallels and of the infinite extension of space, are *a priori* true, and no others conceivable. With those who hold this opinion we have no argument for we meet them on no common ground. We take the standpoint that our knowledge of external space comes to us solely from experience. A system of geometry is an attempt to explain certain facts of experience and, as long as the system is self-consistent, its objective truth or falsity can be judged only by its success in so doing.

We assert: *Any one of the systems of geometry derived from the hypotheses of this article is, as far as our present knowledge goes, in full accord with all facts of experience, provided suitable values are given to the constants involved in each system.*

Experience is necessarily limited to a finite extent of space. Observation and measurement can never extend beyond the most remote astronomical objects, and this distance can well be very small compared with the full extent of objective space. Hence any geometry which agrees with experience only for a restricted portion of space will serve quite as well as Euclid's geometry for explaining the facts of experience. In fact we are not justified in going outside of such a restricted portion of space. Any assumptions, for example, as to the existence or the nature of parallel lines, the connectivity of space, or the displacement of space as a whole, are only justified in case they produce results which would be visible in an arbitrarily small region of space.

The geodesic lines of any space of constant curvature have for a suitably restricted portion of their length all the properties of the "straight lines" of practical life or of Euclidean geometry. In the endeavor to construct a material line which shall be as nearly as possible "straight," we may proceed by attempting to realize the shortest distance between two points; for example, by free-hand drawing, by stretching a string, or by observing the path of a ray of light. The result is simply a geodesic line by definition. Or, we may look for a line which may be revolved upon itself when two points are fixed. This is also a property of a geodesic line in a space of constant curvature, by virtue of the third hypothesis.

A geodesic surface has the properties of a "plane." The practical testing of a plane surface by the application of a "straight" edge has its full significance in a space of constant curvature.

The practical measurement of length and angle gives no criterion as to the curvature of space. Such measurement consists of the assumption of a unit and the application of the unit to the thing measured. These operations are carried out without hindrance in any space of constant curvature. In fact the difference between a space of constant curvature and Euclidean space is not in the manner in which measurement is practically carried out but in the analytic interpretation of the process. So, more generally, any geometric figure may be carried without distortion from one portion of space of constant curvature to another, so that all properties of figures are independent of the positions of the figures in space.

From these considerations it appears that the groundwork of experimental geometry is unaltered by an assumption as to the curvature of space, provided the curvature is constant. Referring to the theorems of the preceding pages, we see that all spaces of constant curvature agree in the first eleven theorems. A distinction appears first in the twelfth theorem. Here we have apparently a means at hand to determine precisely the curvature of external space by measuring the angles of a triangle. The test however fails, owing to the practical impossibility of exact measurements. All we can discover is that the sum of the angles of a triangle does not differ very much from π. And now it is possible to show that if the sides of a triangle in a space of constant curvature, $\pm 1/R^2$, are sufficiently small compared with R the divergence of the sum of its angles from π is within the limits of the errors of observation.* We can say only this: *If the space of experience is of constant curvature* $\pm 1/R^2$, *R must be very large compared with the distance between any two points lying in that part of space in which measurement is possible.*

To complete our argument, we need to ask whether other crucial theorems may not be developed, by means of which the curvature of space may be practically determined. At present, no such theorem has been found. Euclidean geometry in finite space may be derived from our first twelve theorems, and the geometry in a space of constant curvature agrees, as far as has been shown, with Euclidean geometry within the limits of error of observation.

There are still questions to be solved, however, notably in regard to the connectivity of different spaces of constant curvature. It is only within the last twenty-five years that the possibility has been recognized of the existence of such spaces as that represented by the parallelopiped of §8, and of similar forms with other curvature. Consequently many theorems stated in the earlier papers as universally true are now known to be false, or true only in a restricted portion of space. For example, a space or a surface of zero curvature is not necessarily infinite in extent; two "straight" lines in a space of zero curvature may meet in more than one point; and the sum of the angles of a triangle in a space of zero curvature may exceed π, if the surface of the triangle covers a sufficiently large area. It is possible that a more careful study of these spaces may yet reveal properties which render them unfit to explain the facts of experience, but at present no such properties have been found.

---

* See for example the calculation in Lobachevsky's *Zwei geometrische Abhandlungen*, translated by F. Engel, pp. 22–24.

If we are satisfied that the geometries belonging to spaces of constant curvature explain experience, the question naturally arises: *Are these the only systems of geometry which do so?* An answer to this question may be attempted by building up synthetically systems of geometry after the manner of Euclid, but with the assumption of the fewest possible axioms. The most recent example of this mode of procedure is found in the elegant work of Hilbert on the Foundations of Geometry.*

If we maintain the analytic methods of this article, the question demands an examination of our three hypotheses. They have been shown to be sufficient to explain experience; the question now is as to their necessity. As to the first hypothesis it is evident that it must be made, if space is to be treated by the ordinary methods of analysis. With the second hypothesis the case is not so clear, since we have assumed the line element arbitrarily. Helmholtz, however, has obtained this line element as a consequence of certain axioms of motion, which include our third hypothesis. Lie* has shown that Helmholtz's axioms and methods need a thorough revision, but his results remain essentially unaltered. We are, accordingly, justified in saying that no explanation of spatial phenomena has yet been given without the assumption of axioms which lead necessarily to our three hypotheses.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, DECEMBER 1901.

---

* *Grundlagen der Geometrie*, Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen, Teubner, 1899.

* *Cf.* Lie-Engel: *Theorie der Transformationsgruppen*, vol. 3, pp. 437–543. See also Klein, *Lectures on Mathematics, l. c.*